When the measurable Hall theorem fails

Gábor Kun

Rényi Institute+Eötvös University

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Equidecompositions

Γ a set of isometries of $\mathbb{R}^n$. $A, B \subseteq \mathbb{R}^n$ are Γ-equidecomposable if there are finite partitions $A = \cup_{n=1}^k A_n$, $B = \cup_{n=1}^k B_n$ and $\gamma_1, \dots, \gamma_k \in \Gamma$ such that $A_i = \gamma_i B_i$.

Banach, Tarski (1924): Any two bounded sets of nonempty interior in $\mathbb{R}^3$ are equidecomposable. In particular, the unit ball and the union of its two disjoint copies are equidecomposable

The amenable world: circle squaring

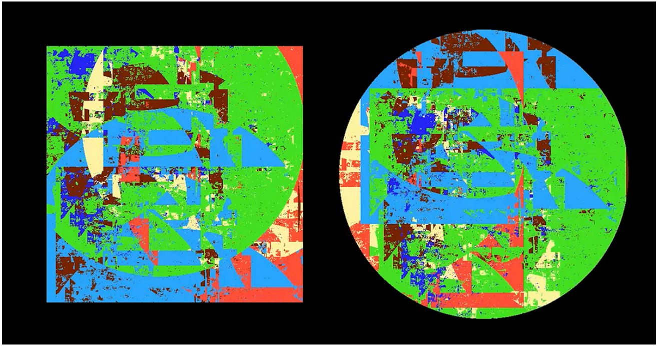
Tarski (1925): Are the unit square and the disc of unit area equidecomposable by isometries?

Laczkovich (1990): Yes! By random translations.

Grabowski, Máthé, Pikhurko (2017): Measurable pieces

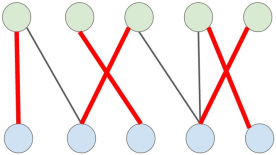
Marks, Unger (2017): Borel equidecomposition

Máthé, Noel, Pikhurko (2021): Boolean combinations of F_σ sets



From equidecompositions to perfect matchings

A perfect matching in a graph G is a set of edges such that every vertex is incident to exactly one of them.



Given a set of isometries Γ, A and B admit a Γ-equidecomposition iff the bipartite graph (=no odd cycle) with vertex set $V(G) = A \cup^* B$, and edge set $E(G) = \{(a, b) : \exists \gamma \in \Gamma \; \gamma a = b\}$ admits a perfect matching Γ.

Hyperfinite graphings

A graphing is a Borel graph over a standard probability measure space whose edge set is the countable union of the graph of measure-preserving bijections.

A graphing is (measurably) hyperfinite if the connectivity relation is hyperfinite (on a conull set), i.e., a countable increasing union of finite relations.

Connes, Feldmann, Weiss (1981): Every pmp group action of an amenable group is measurably hyperfinite.

Fractional perfect matchings and ends

A fractional perfect matching in a graph G is a mapping $\tau : E(G) \to [0,1]$ such that $\sum_{y:(x,y)\in E(G)} \tau((x,y)) = 1$ for every $x \in V(G)$.

Every locally finite hyperfinite graphing with a PM admits a measurable fractional PM!

Adams (1990):Hyperfinite graphings have at most two ends a.e.

A hyperfinite graphing has zero ends a.e. iff the the components are finite, two ends iff it has linear growth a.e. and one end a.e. iff it has superlinear growth. (The growth at vertex x is $r \mapsto |B(x,r)|$.)

Main results on hyperfinite graphings

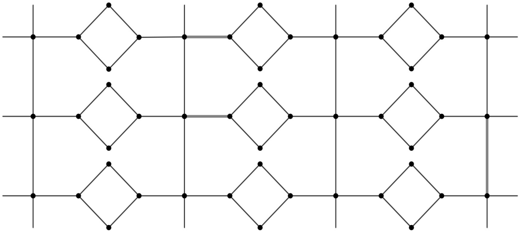
Bowen, K, Sabok '21: A regular hyperfinite bipartite graphing admits a measurable perfect matching if it is one-ended or the degree is odd.

Bowen, K, Sabok '21 : Assume that a hyperfinite bipartite nowhere two-ended graphing G admits a non-integral MFPM τ. Then G admits a MPM.

It is not enough that G is nowhere two-ended, see the next slide.

Why should we look at the MFPM?

Consider a graphing whose orbits are isomorphic to the following graph (can be obtained by the surgery of a pmp free $\mathbb{Z}^2$-action). The choice on certain edges can be forced: 4-cycles should contain two edges of every PM, and no edge connecting a 4-cycle and a vertical line can can be in a PM.



Applications

- Settling the question of Lyons-Nazarov for amenable groups: bipartite Cayley graph has a factor of iid PM iff the group is not the semidirect product of $\mathbb{Z}$ and a finite group of odd size. (Two-ended amenable groups are such semidirect products.)
- Measurable circle squaring: equidecompositions by two independent sets of translations give a measurable equidecomposition.
- "Finally a real application of this abstract nonsense!" Timár (2021): Factor PM of optimal tail between Poisson processes (improving Benjamini-Lyons-Peres-Schramm)
- Measurable balanced orientation in a one-ended graphing

A weaker form of Gardner's conjecture

$\langle \Gamma \rangle$ is the subgroup generated by Γ.

Gardner (1991): $A, B \subseteq \mathbb{R}^n$ bounded measurable, Γ set of isometries. $\langle \Gamma \rangle$ is amenable and A and B are Γ-equidecomposable. Are A and B measurably equidecomposable by isometries?

Laczkovich (1988): Γ is not sufficient.

K (2021): $\langle \Gamma \rangle$ is not sufficient.

Bowen, K, Sabok (2021): True for equidistributed sets unless $\mathbb{Z}_2 * \mathbb{Z}_2 \leq \Gamma$ with finite index

Bowen, K, Sabok (2021): $\Gamma = \Gamma_1 \times \Gamma_2$ amenable, $\Gamma \curvearrowright (X, \nu)$ pmp free action. If two measurable subsets of X are equidecomposable by the induced actions of both Γ_1 and Γ_2, then they are measurably equidecomposable by the action of Γ.

Implies measurable circle squaring if Γ_1, Γ_2 random translations.

A weaker form of Gardner's conjecture II

Consider the Γ_i-equidecompositions and the corresponding bipartite graphings G_i for $i = 1, 2$. These give MFPM's τ_1, τ_2.

We study the graphing G with edge set $supp(\tau_1) \cup supp(\tau_2)$. If a G-component does not contain an infinite G_i-component for $i = 1$ or $i = 2$ then we can find an MPM in it. If it contains infinitely many infinite G_1- or G_2-components then it has superlinear growth.

The union of the set of components containing finitely many but > 0 infinite G_1- and G_2-components both is a nullset: the set of vertices in infinite G_1-components closest to infinite G_2-components is a finite selector.

Expansion in (bipartite) graphings

Assume $\lambda(N(S)) > \varepsilon\lambda(S)$ if $\lambda(S) \leq \frac{1}{2}$.
Includes pmp ergodic actions of Kazhdan Property (T) groups.

Banach-Ruziewicz problem: For $n > 1$ is the only $SO(n)$-invariant finitely additive probability measure on S^n the Lebesgue measure?

Margulis (1980): $n \geq 5$

Sullivan (1981): $n \geq 4$

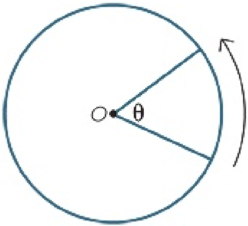
Drinfeld (1984): $n \geq 2$

Lyons, Nazarov (2011): Every bipartite Cayley graph of a non-amenable group admits a factor of iid perfect matching.

Grabowski, Máthé, Pikhurko (2017): $n \geq 3, A, B \subseteq \mathbb{R}^n$ bounded measurable of nonempty interior, $\lambda(A) = \lambda(B)$. Then A and B are measurably equidecomposable.

Graphings without MPM

Laczkovich (1988): 2-regular acyclic graphing without MPM.



Conley, Kechris (2013): Modify it to d-regular for even d.

An (essentially) acyclic graphing is called a treeing.

Marks (2013): d-regular treeing without Borel PM for $d > 2$.

Kechris, Marks (2018): Does every 3-regular graphing admit MPM?

Treeings without MPM and circulation

A circulation is a flow that sums to zero at every vertex.

K (2021): For every $d > 2$ there exists a measurably bipartite, d-regular treeing without circulation. In particular, it has no MPM.

K (2022): For every d there is a free pmp action of $\mathbb{Z}_2^{*d}$ without circulation. No free $\mathbb{Z}$-action on a subset of positive measure and is not the Schreier graphing of a free pmp action of $\mathbb{F}_{d/2}$.

Conjecture (Gurel-Gurevich, Peled): For every probability measure μ on $[0,1]^2$ if $\mu^1 = \mu^2 = \lambda$ and its sections are atomless then its support contains a.e. the graph of a pmp bijection of $[0,1]$.

K (2023): No!

Inverse limit as usual

Consider $\{\varepsilon_n\}_{n=1}^{\infty}$ positive, $\sum_{n=1}^{\infty} \varepsilon_n < \infty$, $\{G_n\}_{n=1}^{\infty}$ finite graphs, $f_n : V(G_{n+1}) \mapsto V(G_n)$. Assume

(V) $\forall n \; \forall S \subseteq V(G_n) \; \left| \frac{|S|}{|V(G_n)|} - \frac{|f_n^{-1}(S)|}{|V(G_{n+1})|} \right| < \varepsilon_n$

(E) $\forall n \; \forall Q \subseteq V(G_n) \; \left| \frac{|Q|}{|E(G_n)|} - \frac{|f_n^{-1}(Q)|}{|E(G_{n+1})|} \right| < \varepsilon_n$

Set $V(\mathcal{G}) = \left\{ (x_n)_{n=1}^{\infty} : \forall n \; x_n \in V(G_n), f_n(x_{n+1}) = x_n \right\}$.
The measure ν defined on cylinder sets $\{(x_n)_{n=1}^{\infty} : x_n \in S\}$ for $S \subseteq V(G_n)$ as $lim_{m \to \infty} \frac{|f_m^{-1} \circ f_{m-1}^{-1} \ldots f_n^{-1}(S)|}{|V(G_{m+1})|}$.

$E(\mathcal{G}) = \{(x, y) : (x_n, y_n) \in E(G_n)$ for all but finitely many $n\}$

G_n d-regular $\to \mathcal{G}$ d-regular a.e.
G_n bipartite, f_n preserves bipartition $\to \mathcal{G}$ measurably bipartite
$E(G_n)$ d-colorable, f_n preserves it $\to E(\mathcal{G})$ d-colorable

How to avoid circulations?

Basic fact: For every circulation c and $p : V(G) \to \mathbb{R}$ bounded $\int_{(x,y)\in E(G)} c(x,y)(p(y) - p(x))d\mu = 0$.

(C) $\forall n$ orientation $\mathcal{O} \in Ori(G_n)$ $\quad \exists p_{\mathcal{O}} : V(G_{n+1}) \to \{1,\ldots,N\}$ s.t. $(p_{\mathcal{O}}(x) - p_{\mathcal{O}}(y)) \in \{-1,0,1\}$ if $(x,y) \in E(G_{n+1})$, and $|\{x : \exists y\ (x,y) \in E(G_{n+1}), p_{\mathcal{O}}(x) - p_{\mathcal{O}}(y) \neq \mathcal{O}(f_n(x), f_n(y))\}| < \varepsilon_n |V(G_{n+1})|$

$(V) + (E) + (C) \to \mathcal{G}$ has no circulation.

Proof: suppose for a contradiction that c is a circulaton in L_1. Approximate $sign(c)$ by $\mathcal{O} \in Ori(G_n)$. Consider the corresponding $p = p_{\mathcal{O}} : V(G_{n+1}) \mapsto \{1,\ldots,N\}$.

Note that $0 = \int c(x,y)(p(y) - p(x))d\mu$ is close to $\int \mathcal{O}(x,y)(p(y) - p(x))d\mu = \|\mathcal{O}\|_1$ minus small loss.

The constructions

Given G and $N \in \mathbb{N}$ define $G' = G'(G, N)$ as

$V(G') = V(G) \times \Pi_{\mathcal{O} \in Ori(G)}\{1, \ldots, N\}$ and $E(G') =$
{((x,v), (x',v')): (x,x') $\in E(G), \forall \mathcal{O} \in Ori(G)\ v'_{\mathcal{O}} - v_{\mathcal{O}} = \mathcal{O}(x, x')\}$.

This is regular but on the boundary.

Start with $G_1 = K_{d,d}$. Define G_{n+1}:

To prove the $\mathbb{Z}_2 * \mathbb{Z}_2$ theorem take two copies of $G'(G_n, N)$ and connect twins with degree less than d (possible multiple edges).

For the treeing without MPM take d copies and add $(d - degree)$ vertices for every k tuple of siblings connected to them.

For the non-sparse theorem double the previous one in every step.

Thank you!